Vol. 3 Issue No.3, July-September 2021 e-ISSN 2456-7701

Journal of Science and Technological Researches

A Peer Reviewed Journal

Origin of Innovation

Domain: www.jstr.org.in, Email: editor@jstr.org.in

SOME FIXED POINT THEOREM FOR ASYMPTOTICALLY REGULAR MAPS IN N-FUZZY METRIC SPACE

Shikha Shende[1]*, Geeta Agrawal[1] and Neeraj Malviya[2]

[1]Department of Mathematics, Govt. M.V.M. Bhopal, M.P., India.

[2]Department of Mathematics, Govt. College Timarni, Harda, M.P., India

Email: shikhashende1701@gmail.com

Date of Received	**09 June, 2021**
Date of Revised	**19 June, 2021**
Date of Acceptance	**26 June, 2021**
Date of Publication	**30 September, 2021**

DOI : https://doi.org/10.51514/JSTR.3.3.2021.9-13

e-ISSN 2456-7701
Vol. 3 Issue No.3, July-September 2021

SOME FIXED POINT THEOREM FOR ASYMPTOTICALLY REGULAR MAPS IN N-FUZZY METRIC SPACE

Shikha Shende[1]*, Geeta Agrawal[1] and Neeraj Malviya[2]

[1]Department of Mathematics, Govt. M.V.M. Bhopal, M.P., India.

[2]Department of Mathematics, Govt. College Timarni, Harda, M.P., India.

Email: shikhashende1701@gmail.com

ABSTRACT

In this paper, we investigate two fixed point theorem on the structure of N-fuzzy metric space using asymptotically regular map and asymptotically regular sequence. This result generalizes the paper of Goswami et. al. [5].

Keywords: *N- fuzzy metric space, fixed point theorem, asymptotically regular maps, asymptotically regul ar sequence.*

INTRODUCTION

In 1975, Kramosil and Michalek [6] introduced the concept of fuzzy metric space. George and Veeramani [4] modified the concept of fuzzy metric space.In 1963, Gahler [2, 3] generalized usual notion of metric space called 2-metric space. Using the notion of 2-metric space, S. Sharma [14] and S. Kumar [7] introduced fuzzy-2-metric spaces without knowing each other but Ha et al. shows that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings. In 1992 Bapure Dhage [1] in his Ph.D thesis introduced a new class of generalized metric space called D-metric space [8]. B. Singh and M. Chouhan [15] defined S-fuzzy metric space by using the concept of D-metric space. However, Mustafa and Sims in [9] have pointed out that most of the results claimed by Dhage and others in D-metric spaces are invalid. To overcome these fundamental flaws, they introduced a new concept of generalized metric space called G-metric space [9]. Using the concept of G-metric space, G. Sun and K Yang [16] introduced the notion of Q-Fuzzy metric space, K.P.R. Rao et. al. [10] proved two fixed point theorems in symmetric Q(G)-metric space, Sedghi et. al. in [13] introduced D*-metric space whichis a generalization of G-metric space and gave an example which is D* metric space but not G-metric space. Using the concept of D*-metric space, Sedghi and Shobe [11] defined M-fuzzy metric space. Very recently, Sedghi et. al [12] defined S-metric space which is a generalization of D*-metric space and G-metric space and justified their work by various examples and definitions related to topology of S-metric space. N. Malviya [17] introduced the notion of N-fuzzy metric space, pseudo N-fuzzy metric space and describe some of their properties and examples.

In this paper, using the notion of asymptotic regularity of mapping, we prove some fixed point theorem in N-fuzzy metric space.

Preliminaries

Definition 1:- A mapping $*: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if ([0,1], $*$) is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ for $a \leq c, b \leq d$. Examples of t-norms are $a * b = \min\{a, b\}$, $a * b = ab$ and $a * b = \max\{a + b - 1, 0\}$

Definition 2:- A triplet $(X, N, *)$ is an N-fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and N is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $r, s, t > 0$

(1) $N(x, y, z, t) > 0$

(2) $N(x, y, z, t) = 1$ if and only if $x = y = z$

(3) $N(x, y, z, r + s + t) \geq N(x, x, a, r) * N(y, y, a, s) * N(z, z, a, t)$

(4) $N(x, y, z, .): (0, \infty) \rightarrow (0, 1)$ is a continuous function.

Definition 3:- let $(X, N, *)$ be a N-fuzzy metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is a Cauchy sequence in X if ∀ ε ∈ (0, 1), ∃ n_0∈ N such that N(x_n, x_n, x_m, t) > 1−ε, ∀ n, m ≥ n_0, or equivalently, if $\lim_{n,m\rightarrow\infty}$ N(x_n, x_n, x_m, t) = 1, ∀ t > 0. $\{x_n\}$ converges to x if ∀ ε ∈ (0, 1), ∃ n_0 ∈ N such that $N(x, y, z, t) >$ 1−∈ ∀ n ≥ n_0.

X is said to be complete if and only if every

*Author for correspondence

cauchy sequence converges in X.

Definition 4:- A mapping $\emptyset : [0,1] \to [0,1]$ is called an altering distance function if

(i) φ is strictly decreasing and left continuous.

(ii) φ(λ) = 0 if and only if λ = 1

i.e, $\lim_{\emptyset \to 1-} \emptyset(1) = 0.$

Definition 5:- Let f and g be self mappings on a N-fuzzy metric space (X, N, *) and $\{x_n\}$ be a sequence in X. f is said to be asymptotically regular at a point $x_0 \in X$ if

$\left(\lim_{n\to\infty} N(p^n(x_0),\ p^n(x_0),\ p^{n+1}(x_0), t\right) = 1, \forall\, t > 0.$

Also the sequence $\{x_n\}$ is said to be asymptotically regular with respect to the pair (p, q) if $\lim_{n\to\infty} N(p(x_n), p(x_n), q(x_n), t) = 1, \forall\, t > 0.$

Definition 6:- Two self-mapping f and g on a fuzzy metric space (X, N, *) are said to be compatible if

$\lim_{n\to\infty} N(pq(x_n), pq(x_n), qp(x_n), t) = 1, \ \forall\ t > 0,$ where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} q(x_n) = x$, for some x ∈ X.

Theorem 1:- Let (X, N, *) be a complete fuzzy metric space, $\emptyset$ be the altering distance function and p : X → X be such that the following condition is satisfied:

$$\emptyset(N(p(x),p(x),p(y),t)) \le b_1(x,y)\theta[\min\{\emptyset(N(x,x,p(x),t)),\emptyset(N(y,y,p(y),t))\}] + b_2(x,y)\varphi[\emptyset(N(x,x,p(x),t)).\emptyset(N(y,y,p(y),t))] + b_3(x,y)\emptyset(N(x,x,y,t)) + b_4(x,y)\left(\emptyset(N(x,x,p(x),t)) + \emptyset(N(y,y,p(y),t))\right) + b_5(x,y)[\emptyset(N(x,x,p(y),t)) + \emptyset(N(p(x),p(x),y,t))] \quad \textbf{...(1)}$$

$\forall\, x,y \in X, t > 0$ where $b_i : X \times X \to [0,\infty),\ i = 1,2,3,4,5$ *are such that* for some arbitrarily fixed $\lambda_1 > 0, 0 < \lambda \quad 0 < \lambda_2 < 1$

$$b_1(x,y)+b_2(x,y) \le \lambda_1$$

$$b_3(x,y)+b_4(x,y)+2b_5(x,y) \le \lambda_2$$

And $\theta, \psi : R^+ \to R^+$ are continuous functions at and $\theta(0) = \psi(0) = 0.$

If s is asymptotically regular at some point $x_0 \in X$, then p has a unique fixed point in X.

Proof. Suppose that $\{x_n\}$ is a sequence in X where $x_0 \in X$ and $x_{n+1} = p(x)\ \forall\ n \ge 0$, Now if for some n ≥ 0, $x_n = x_{n+1}$, then x_n is a fixed point of f . Suppose that $x_n \ne x_{n+1} \forall\, n$. We show that the sequence $\{x_n\}$ is Cauchy.

Suppose to the contrary $\exists\, 0 < \in < 1, t > 0$ and two sequence of integers $\{r_n\}$ and $\{s_n\}$ such that $r_n > s_n > n,$

$$N(x_{r_n}, x_{r_n}, x_{s_n}, t) \le 1-\in$$
$$N(x_{r_n-1}, x_{r_n-1}, x_{s_n-1}, t) > 1-\in$$
$$N(x_{r_n-1}, x_{r_n-1}, x_{s_n}, t) > 1-\in,\ \forall\, n \in \mathbb{N} \cup \{0\} \quad \textbf{...(4)}$$

Now we have

$$1-\in \ge N(x_{r_n}, x_{r_n}, x_{s_n}, t) \ge N\left(x_{r_n}, x_{r_n}, x_{r_n-1}, \frac{t}{3}\right) * N\left(x_{r_n}, x_{r_n}, x_{r_n-1}, \frac{t}{3}\right) * N\left(x_{s_n}, x_{s_n}, x_{r_n-1}, \frac{t}{3}\right)$$

$$\Rightarrow 1-\in \ge \lim_{n\to\infty} N(x_{r_n}, x_{r_n}, x_{s_n}, t) \ge (1 * 1 * 1-\in)$$

(since f asymptotically regular at x_0)

$$\Rightarrow \lim_{n\to\infty} N(x_{r_n}, x_{r_n}, x_{s_n}, t) = 1-\in \quad - \quad \textbf{...(5)}$$

Again,

$$N(x_{r_n}, x_{r_n}, x_{s_n-1}, t) \ge N\left(x_{r_n}, x_{r_n}, x_{s_n}, \frac{t}{3}\right) * N\left(x_{r_n}, x_{r_n}, x_{s_n}, \frac{t}{3}\right) * N\left(x_{s_n-1}, x_{s_n-1}, x_{s_n}, \frac{t}{3}\right)$$

$$\Rightarrow \lim_{n\to\infty} N(x_{r_n}, x_{r_n}, x_{s_n-1}, t) > 1-\in \quad \textbf{...(6)}$$

Taking $x = x_{r_n-1}$ and $y = x_{s_n-1}$ in (1), we have

$$\emptyset\left(N(x_{r_n}, x_{r_n}, x_{s_n}, t)\right) \le b_1(x,y)\theta\left(\min\left\{\emptyset\left(N(x_{r_n-1}, x_{r_n-1}, x_{r_n}, t)\right), \emptyset\left(N(x_{s_n-1}, x_{s_n-1}, x_{s_n}, t)\right)\right\}\right) + b_2(x,y)\varphi\left(\emptyset\left(N(x_{r_n-1}, x_{r_n-1}, x_{r_n}, t)\right).\emptyset\left(N(x_{s_n-1}, x_{s_n-1}, x_{s_n}, t)\right)\right) + b_3(x,y)\emptyset\left(N(x_{r_n-1}, x_{r_n-1}, x_{s_n-1}, t)\right) + b_4(x,y)\left[\emptyset\left(N(x_{r_n-1}, x_{r_n-1}, x_{r_n}, t)\right) + \emptyset\left(N(x_{s_n-1}, x_{s_n-1}, x_{s_n}, t)\right)\right] + b_5(x,y)\left[\emptyset\left(N(x_{r_n-1}, x_{r_n-1}, x_{s_n}, t)\right) + \emptyset\left(N(x_{r_n}, x_{r_n}, x_{s_n-1}, t)\right)\right]$$

Taking $n \to \infty$ and by (4), (5), (6) and using the fact that p is asymptotically regular at x_0 we have ,

$$\emptyset(1\text{-}\in) \le b_3(x,y)\emptyset(1\text{-}\in) + 2b_5(x,y)\emptyset(1\text{-}\in) < \emptyset(1\text{-}\in)$$

Which is a contradiction.

Thus $\{x_n\}$ is a Cauchy sequence. Since (X, N, *) is a complete N-fuzzy metric space, $\exists\, z \in X$ such that $x_n \to z$.

Now,

$$\emptyset\left(N(p(x_n), (p(x_n), p(z), t)\right) \le b_1(x,y)\theta(\min\{\emptyset(N(x_n, x_n, x_{n+1}, t)), \emptyset(N(z,z,p(z),t))\}) + b_2(x,y)\varphi\left(\emptyset(N(x_n, x_n, x_{n+1}, t)), \emptyset(N(z,z,p(z),t))\right) + b_3(x,y)\emptyset(N(x_n, x_n, z, t)) +$$

e-ISSN 2456-7701
Vol. 3 Issue No.3, July-September 2021

$b_4(x,y)[\emptyset(N(x_n,x_n,x_{n+1},t)) + \emptyset(N(z,z,p(z),t))] +$
$b_5(x,y)[\emptyset(N(x_n,x_n,p(z),t)) + \emptyset(N(z,z,x_{n+1},t))]$

For $n \to \infty$

$$\lim_{n\to\infty} \emptyset(N(z,z,p(z),t)) \le [\,b_4(x,y) + b_5(x,y)] \lim_{n\to\infty} \emptyset(N(z,z,p(z),t)\,)$$

$$\Rightarrow [1 - b_4(x,y) - b_5(x,y)] \lim_{n\to\infty} \emptyset(N(z,z,p(z),t)) \le 0$$

$$\Rightarrow \lim_{n\to\infty} \emptyset(N(z,z,p(z),t)) = 0$$

(Since $0 < b_3(x,y) + b_4(x,y) + 2b_5(x,y) < 1$)

$$\Rightarrow p(z) = z.$$

If u is another fixed point of f in X, then

$$\emptyset(N,p(u),p(u),p(z),t))$$
$$\le b_1(x,y)\theta(\min\{\emptyset(N(u,u,p(u),t)),\emptyset(N(z,z,p(z),t))\})$$
$$+ b_2(x,y)\varphi(\emptyset(N(u,u,p(u),t)),\emptyset(N(z,z,p(z),t)))$$
$$+ b_3(x,y)(\emptyset(N(u,u,z,t))$$
$$+ b_4(x,y)[\emptyset(N(u,u,p(u),t)) + \emptyset(N(z,z,p(z),t))]$$
$$+ b_5(x,y)\ [(\emptyset(N(u,u,p(z),t)) + \emptyset(N(z,z,p(u),t))]$$
$$\Rightarrow \emptyset(N(u,u,z,t)) \le b_3(x,y)\emptyset(N(u,u,z,t)) + 2b_5(x,y)\emptyset(N(u,u,z,t))$$
$$\Rightarrow [1 - b_3(x,y) - 2b_5(x,y)]\ \emptyset(N(u,u,z,t)) \le 0$$
$$\Rightarrow \emptyset(N(u,u,z,t)) = 0$$

(Since $0 < b_3(x,y) + b_4(x,y) + 2b_5(x,y) < 1$)

$$\Rightarrow u = z,$$

Example:- Let (x, N, *) is N-fuzzy metric space where $X = [0,1]$ and N is a fuzzy set on $X^3 \times (0,\infty)\ to\ [0,1]$ defined by

$$N(x,y,z,t) = \frac{t}{t + [\,|x-z| + |y-z|\,]}$$

For all $x,y,z \in X$ and "*" be the minimum t-norm (as defined in definition) define $S: X \to X$ by

$$S(x) = \begin{cases} \frac{x}{3}, & x \in [0,\frac{1}{2}] \\ \frac{1}{6}, & x \in [\frac{1}{2},1] \end{cases}$$

and $\emptyset(\lambda) = 1 - \lambda,\ \lambda \in [0,1]$

Let $\phi(p) = \sqrt{p}$ and $\psi(q) = q^2, \forall\, p,q \in R^+$

Also let $b_1(x,y) = |x-y|,\ b_2(x,y) = |x^2\text{-}y^2|$

$$b_3(x,y) = \begin{cases} \frac{1}{|x-y|}, & x \ne y \\ 0, & x = y \end{cases}$$

$$b_4(x,y) = b_5(x,y) = |\frac{1 - b_3(x,y)}{3}|$$

Then all conditions of theorem 1 satisfied, hence by theorem 1 δ has unique fixed point 0 in X.

Corollary:- Let p, q : X → X be mapping on a complete fuzzy metric space (X, N, *) and ∅ be the altering distance function. Let p and q be asymptotically regular at a point Type equation here. x0 ∈ X and both satisfy the inequality (1). Moreover, if

$$\emptyset(N(p(x),p(x),q(y),t)) \le k\left(\emptyset(N(x,x,y,t)) + \emptyset(N(x,x,p(x),t)) + \emptyset(N(y,y,q(y),t))\right), \quad \textbf{...(7)}$$

Where $0 < k < 1\ and\ x,y \in X,$

Then p and q have a unique common fixed point in X.

Proof. From theorem [1], both f and g have unique fixed points say, z and v respectively.

Since f and g satisfy (7),

$$\emptyset(N(p(z),p(z),q(v),t)) \le\ k\left(\emptyset(N(z,z,v,t)) + \emptyset(N(z,z,p(z),t)) + \emptyset(N(v,v,q(v),t))\right),$$

$$\Rightarrow (N(z,z,v,t)) \le k\left(\emptyset(N(z,z,v,t)) + \emptyset(N(z,z,z,t)) + \emptyset(N(v,v,v,t))\right)$$

$$\Rightarrow (1-k)\emptyset(N(z,z,v,t)) \le 0$$

$$\Rightarrow \emptyset(N(z,z,v,t)) \le 0 \qquad (since\ k < 1)$$

$$\Rightarrow z = v$$

i.e., p and q have a unique common fixed point.

Theorem 2:- Let $p: X \to X$ be a mapping on a complete fuzzy metric space (X, N, *) and φ is the altering distance function. If f is asymptotically regular at a point x_0 ∈ X and f satisfies,

$\emptyset$(N(f(x),f(x),f(y),t)) ≤h_1 min
{∅(N(x,x,y,t)),∅(N(f(x),f(x),x,t)),∅(N(f(x),f(x),y,t))}
+h_2 min{∅(N(x,x,y,t)),∅(N(f(y),f(y),y,t)),∅(N(x,x,f(y),t))}

For all x, y ∈ X, $t > 0$, where h_1, $h_2 > 0$ are constants such that $h_1 + h_2 < 1$,

Then f has a unique fixed point in X.

Proof. As in Theorem 1, we construct a sequence $\{x_n\}$ in X by $x_{n+1} = f(x_n)$ ∀ n ∈ ℕ ∪ {0}, where x_0 ∈ X. If there exists n with $x_n = x_{n+1}$, then x_n is a fixed point of f. Suppose that $x_n \ne x_{n+1}$ for all n.

To show that $\{x_n\}$ is a Cauchy sequence.

Let m, n ∈ ℕ ∪ {0}. From (8)

$$\emptyset(N(p(x\_n),p(x\_n),p(x\_m),t))$$
$$\le h\_1\ \min\{\emptyset(N(x\_n,x\_n,x\_m,t)),\emptyset(N(p(x\_n\ ),p(x\_n\ ),x\_n,t)),\emptyset(N(p(x\_n\ ),p(x\_n\ ),x\_m,t))\}$$
$$+ h\_2\ mi\ \text{n}\{\emptyset(N(x\_n,x\_n,x\_m,t)),\emptyset(N(p(x\_m\ ),p(x\_m\ ),x\_m,t)),\emptyset(N(x\_n,x\_n,\ 〖p(x〗\_m),t)\}$$

Since, f is asymptotically regular at $x_0 \in X$, taking n, m → ∞.

$$\lim_{n,m\to\infty} \emptyset\big(N(p(x_n), p(x_n), p(x_m), t)\big) = 0$$

$$\Rightarrow \lim_{n,m\to\infty} N(p(x_n), p(x_n), p(x_m), t) = 1.$$

i.e. $\{x_n\}$ is a Cauchy sequence in (X, N, *) since (X, N, *) is complete, therefore $x_n \to z$(say) n X.

Using (8)

$$\emptyset\big(N(x_{n+1}, x_{n+1}, p(z), t)\big)$$
$$= \emptyset\big(N(p(x_n), p(x_n), p(z), t)\big) \leq$$
$$h_1 \min\{\emptyset\big(N(x_n, x_n, z, t)\big), \emptyset\big(N(p(x_n), p(x_n), x_n, t)\big),$$
$$\big(N(p(x_n), p(x_n), z, t)\big)$$
$$+ h_2 \min\{\emptyset\big(N(x_n, x_n, z, t)\big), \emptyset\big(N(p(z), p(z), z, t)\big),$$
$$\emptyset\big(N(x_n, x_n, p(z), t)\big)\}$$
$$\Rightarrow \emptyset\big(N(z, z, p(z), t)\big) = 0$$

$\Rightarrow p(z) = z,$ establishes that z is a fixed point for p.

Uniqueness can be shown easily.

Hence, z is the unique fixed point of p.

Theorem 3 :- Let (X, N, *) be a fuzzy metric space, ∅ be the altering distance function and p and q be two commutative self-mappings on X such that

$$\emptyset\big(N(p(x), p(x), p(y), t)\big) \leq$$
$$k_1[\emptyset\big(N(q(x), q(x), q(y), t)\big) +$$
$$k_2\left(\emptyset\big(N(q(x), q(x), p(y), t)\big) +\right.$$
$$\left.\emptyset\big(N(q(y), q(y), p(y), t)\big)\right]$$

...(9)

Where x, y $\in X, t > 0$ and $k_1 : \mathbb{R} \to [0,1), 0 < k_1,\ k_2 < 1.$

Moreover if

(i)

and q are asymptotically regular at x_0.

(ii)

(X) ⊆ q(X),

(iii)

(X) or q(X) is a complete subspace of X ,

then p and q have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $p(X) \subseteq q(X)$, define a sequence $\{u_n\}$ by $u_{n+1} = p(x_n) = q(x_{n+1}),\quad n \in \mathbb{N} \cup \{0\}$. Again since p and q are asymptotically regular at x_0,

$$\lim_{n\to\infty} \emptyset(N(u_n, u_n, u_{n+1}, t)) = 0 \qquad \textbf{...(10)}$$

To show that the sequence $\{u_n\}$ is cauchy.

Suppose, theres exists $0 < \in < 1$ and two sequence of integers $\{r_n\}$ *and* $\{s_n\}$ *such that* $r_n > s_n > n$,

$$N(u_{r_n}, u_{r_n}, u_{s_n}, t) \leq 1 - \epsilon,$$
$$N(u_{r_n-1}, u_{r_n-1}, u_{s_n-1}, t) > 1{-}\in,$$
$$N(u_{r_n-1}, u_{r_n-1}, u_{s_n}, t) > 1{-}\in, \forall n \in N \cup \{0\}.$$

...(11)

Following the technique applied in theorem 2.1 we can show that

$$\lim_{n\to\infty} N\big(u_{r_n}, u_{r_n}, u_{s_n}, t\big) = 1{-}\in,\ t > 0$$

...(12)

$$\emptyset\left(N\left(u_{r_n+1}, u_{r_n+1}, u_{s_n+1}, t\right)\right) =$$
$$\emptyset\left(N(p(x_{r_n}), p(x_{r_n}), p(x_{s_n}), t)\right) \leq$$
$$k_1[\emptyset(N\left(q(x_{r_n}), q(x_{r_n}), q(x_{s_n})\right) +$$
$$k_2\left(\emptyset\left(N(q(x_{r_n}), q(x_{r_n}), p(x_{r_n}), t)\right) +\right.$$
$$\left.\emptyset\left(N\left((q(x_{s_n}), q(x_{s_n}), p(x_{s_n}), t)\right)\right)\right]$$

Taking $n \to \infty$ and using (10) and (12) we have

$$\emptyset(1{-}\in) \leq k_1\emptyset(1{-}\in) < \emptyset(1{-}\in)$$

a contradiction. Hence $\{u_n\}$ is Cauchy sequence. Suppose that $q(X)$ is complete, then there exist $v \in q(X)$ such that

$\lim_{n\to\infty} u_n = v$. Also, for some $z \in X$ we have $q(z) = v$.

Now,

$$\emptyset\big(N(p(z), p(z), u_{n+1}, t)\big)$$
$$= \emptyset\big(N(p(z), p(z), p(x_n), t)\big)$$
$$\leq k_1\Big[\emptyset\big(N(q(z), q(z), q(x_n), t)\big)$$
$$+ k_2\Big(\emptyset\big(N(p(z), p(z), q(z), t)\big)$$
$$+ \emptyset\big(N(p(x_n), p(x_n), q(x_n), t)\big)\Big)\Big]$$

For $n \to \infty$,

$$\emptyset\big(N(p(z), p(z), v, t)\big) \qquad p$$
$$\leq k_1\big[k_2\emptyset\big(N(p(z), p(z), v, t)\big)\big]$$
$$\Rightarrow (1 - k_1.k_2)\emptyset\big(N(p(z), p(z), v, t)\big) = 0 \qquad p$$
$$\Rightarrow \emptyset\big(N(p(z), p(z), v, t)\big) = 0 \qquad p$$
$$\Rightarrow p(z) = v$$

Therefore $p(z) = v = q(z)$ i.e., z is the coincident point of p and q.

Next, from (9),

$$\emptyset\left(N(p(p(z)), p(p(z)), p(z), t)\right)$$
$$\leq k_1\left[\emptyset\left(N(q(p(z)), q(p(z)), q(z), t)\right)\right.$$
$$+ k_2\left(\emptyset\left(N(p(p(z)), p(p(z)), q(p(z)), t)\right)\right.$$
$$\left.\left.+ \emptyset\big(N(p(z), p(z), q(z), t)\big)\right)\right]$$

$$= k_1 \Big[\emptyset\Big(N\big(p(q(z)), p(q(z)), q(z), t\big)\Big) + k_2\Big(\emptyset\Big(N\big(p(p(z)), p(p(z)), p(q(z)), t\big)\Big) + \emptyset\big(N(p(z), p(z), q(z), t)\big)\Big)\Big]$$

$(since\ \ pq = qp)$

$$= k_1 \Big[\emptyset\Big(N\big(p(p(z)), p(p(z)), p(z), t\big)\Big) + k_2\Big(\emptyset\Big(N\big(p(p(z)), p(p(z)), p(p(z)), t\big)\Big) + \emptyset\big(N(p(z), p(z), p(z), t)\big)\Big)\Big]$$

$$= k_1 \emptyset\Big(N\big(p(p(z)), p(p(z)), p(z), t\big)\Big)$$

$$\Rightarrow (1 - k_1)\emptyset\Big(N\big(p(p(z)), p(p(z)), p(z), t\big)\Big) = 0$$

$$\Rightarrow \emptyset\Big(N\big(p(p(z)), p(p(z)), p(z), t\big)\Big) = 0$$

$$\Rightarrow p(p(z)) = p(z) = v$$

Similarly, $q\big(q(z)\big) = q(z) = v.$

Hence v is a common fixed point of p and q. If v_1 is another common point fixed point of p and q, then

$$\emptyset\big(N(p(v), p(v), p(v_1), t)\big) \le k_1 \Big[\emptyset\big(N(q(v), q(v), q(v_1), t)\big) + k_2\Big(\emptyset\big(N(p(v), p(v), q(v), t)\big) + \emptyset\big(N(p(v_1), p(v_1), q(v_1), t)\big)\Big)\Big]$$

$$\Rightarrow \emptyset\big(N(v, v, v_1, t)\big) \le k_1 \Big[\emptyset\big(N(v, v, v_1, t)\big) + k_2\Big(\emptyset\big(N(v, v, v, t)\big) + \emptyset\big(N(v_1, v_1, v_1, t)\big)\Big)\Big]$$

$$\Rightarrow (1 - k_1)\emptyset\big(N(v, v, v_1, t)\big) = 0$$

$$\Rightarrow v = v_1$$

Hence p and q have a unique common fixed point in X.

REFERENCES

[1]. Dhage B.C., Generalized metric spaces and mapping with fixed points, Bulletin of the Calcutta Mathematical Society, 84(1992), 329-336.

[2]. Gahlers S., 2-metrische Raume and ihre topologische structure, Math.Nachr, 26(1963), 115-148.

[3]. Gahlers S., Zur geometric 2-metrische raume, Revue Roumaine Math.Pures Appl., 11(1966), 665-667

[4]. George A., Veeramani P., On some results in fuzzy metric spaces, Fuzzy Sets Syst., 64(1994), 395-399.

[5]. Goswami N., Patir B. Fixed point theorems for asymptotically regular mappings in fuzzy metric Spaces, Korean J Math 27 (2019), No.4, pp 861-877

[6]. Kramosil O., Michalek J., Fuzzy metrics and statistical metric spaces, Kybernetika, 11(1975), 326-334.

[7]. Kumar S., Common fixed point theorem in fuzzy 2-metric spaces, Uni.Din. Bacau. Studii Si Cercetiri Sciintifice, Serial: Mathematical, 18(2008), 111-116.

[8]. Mustafa Z., Sims B., Some remarks concerning D-metric space, International Conference on Fixed Point Theory and Applications, Valencia, Spain, (2003), 189-198.

[9]. Mustafa Z., Sims B., A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis, 7(2) (2006), 289-297.

[10]. Rao K.P.R., Altun I., Hina Bindu S., Common coupled fixed point theorems in generalized fuzzy metric spaces, Advances in Fuzzy Systems, 2011 Article ID 986748, 6 Pages

[11]. Sedghi S., Shobe N., A common fixed point theorem in two M-fuzzy metric space, Commun, Korean. Math. Soc., 22(4) (2007), 513-526.

[12]. Sedghi S., Shobe N., Aliouche A., A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64(3) (2012), 258-266

[13]. Sedghi S., Shobe N., Zhou H., A common fixed point theorem in D∗-metric spaces, Fixed Point Theory and Application, (2007), Article ID 27906, 13 pages.

[14]. Sharma S., On fuzzy metric space, Southeast Asian Bulletin of Mathematics, 26(2002), 133-145.

[15]. Singh B., Chauhan M.S., Generalized fuzzy metric spaces and fixed point theorems, Bull, Cal. Math. Soc., 89(1997), 457- 460.

[16]. Sun G., Yang K., Generalized fuzzy metric spaces with properties, Research journal of Applied Sciences, Engineering and Technology, 2(7) (2010), 673-678.

[17]. Malviya N., The N-fuzzy metric space and mappings with application DOI:10.1515/ fascmath 2015-0019.